Nova Série
BOLETIM
DA SOCIEDADE BRASILEIRA DE MATEMÁTICA

Bol. Soc. Bras. Mat., Vol.31, No. 1, 29–43
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Smooth classification of 1-resonant vector fields on $\mathbb{R}^3$

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Abstract. In this paper we study on $\mathbb{R}^3$ a class of smoothly (C^∞) finitely determined vector fields which admit infinite many resonant relations. We give a complete classification of all such vector fields with arbitrarily degenerated nonlinear parts.

Keywords: finite determinacye, normal form, resonance, vector fields.

Mathematical subject classification: 34C20, 58F36

1 Introduction

Let $\mathbf{Q}$ be a set of germs of vector fields on $\mathbb{R}^n$ at a singular point 0. What is the simplest normal form to which any vector field of $\mathbf{Q}$ can be reduced by a change of coordinates? This is a classical question and, in the case that $\mathbf{Q}$ consists of all vector fields with a fixed linear approximation it is answered by the Poincare-Dulac-Belitskii theorem which says that under a formal change of coordinates any vector field with a fixed linear part $\dot{x} = Jx$ can be reduced to the resonant normal form. Namely, assuming that the matrix J has the Jordan normal form, $J = S + N$, where S is the semi-simple part of J and N the nilpotent part, one can reduce the given system, via a formal change of coordinates, to a system of the form $\dot{y} = Jy + h(y)$, where the formal series $h(y) = (h_1(y), \dots, h_n(y))^t$ satisfies the following relations.

$$Sh(y) - h'(y)Sy = 0, \quad N^t h(y) - h'(y)N^t y = 0. \tag{1}$$

Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of J, then the first equality of (1) means that $h_i(y) = \sum_{|\alpha|\geq 2} h_{i,\alpha} y^\alpha$, where the coefficients $h_{i,\alpha} = 0$ if $\lambda_i \neq (\alpha, \lambda)$. In

Received 19 August 1999.

other words, $h(y)$ only contains *resonant monomials* $y^\alpha \partial_{y_i}$ which correspond to *resonant relations*

$$\lambda_i = (\alpha, \lambda), \quad |\alpha| \geq 2. \tag{2}$$

In the above we adopt the usual multi-index notations: $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_j \in \mathbb{Z}_+$, the set of non-negative integers, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $(\alpha, \lambda) = \alpha_1\lambda_1 + \cdots + \alpha_n\lambda_n$, $y^\alpha = y_1^{\alpha_1} \cdots y_n^{\alpha_n}$. We also use ∂_{y_i} to stand for $\frac{\partial}{\partial y_i}$.

It is clear that if there are a finite number of resonant relations between eigenvalues of the matrix J then any smooth vector field $\dot{x} = Jx + \cdots$ is formally reducible to a polynomial. Since such vector fields are necessarily hyperbolic, therefore, by the Chen theorem, the normalization can also be in smooth category. If a vector field admits infinitely many resonant relations between the eigenvalues, then, in the formal category, its finite determinacy is characterized by the Ichikawa theorem (see shortly afterwards). The Ichikawa theorem implies that such vector fields are either hyperbolic (the inverse statement that hyperbolic vector fields are finitely determined is obviously false) or partially hyperbolic. In the former case, again due to the Chen theorem, the normalization can be taken smooth, while in the latter case, the center manifolds of the vector fields have to be 1-dimensional or 2-dimensional with $\pm\lambda i$ as eigenvalues. To these quasi-hyperbolic vector fields one can apply the Belitskii theorem to insure the smoothness of the normalization. More exactly, in [1] Belitskii generalizes the Chen and the Ichikawa theorems to the C^∞ quasi-hyperbolic category: a smooth local vector field is C^∞ k-determined if and only if it is formally k-determined. Moreover, it is proved in [1] that if a smooth vector field X is formally finitely determined (i.e., k-determined for some $k < \infty$) and a smooth vector field Y is formally conjugated to X, then Y is C^∞ conjugated to X. This result implies that the C^∞- and the formal k-determinacy are the same property, and thus it reduces the C^∞ classification of finitely determined vector fields to the formal category.

The set of all formally finitely determined vector fields is characterized by the Ichikawa theorem (see [4, 5]), which, due to the Belitskii theorem as mentioned above, coincides with the set of all smoothly finitely determined vector fields. To describe this set, we introduce the following definition

Definition 1.1. *A vector field with eigenvalues λ is called l-resonant if the number of generators of the semigroup $\{(\lambda, \alpha) = 0,\ \alpha \in \mathbb{Z}_+^n\}$ is l.*

The following statements are from [4, 5, 1].

Proposition 1.1. *A smooth vector field admitting infinitely many resonant relations is smoothly finitely determined if and only if it is 1-resonant and the* *nonlinear part does not belong to a certain exceptional set* **E** *of infinite codimension in the space of all vector functions.*

Remark 1.1. *The exceptional set* **E** *in the proposition can be described in terms of the linear approximation. In fact, let X be a 1-resonant vector field with the Poincare-Dulac normal form $\dot{x}_j = \lambda_j x_j + x_j f_j(x^\alpha) + \tilde{x} g_j(x^\alpha)$, where $(\lambda, \alpha) = 0$ and $\tilde{x} g_j(x^\alpha)$ are possible resonant terms when extra resonant relations arise (see non-strongly 1-resonant case shortly), then the exceptional set E consists of those vector fields with $(f(x^\alpha), \alpha) = 0$, where $f = (f_1, \ldots, f_n)$.*

Proposition 1.1 says that, although the Poincare-Dulac normal form of a 1-resonant vector field consists of infinitely many resonant monomials, the vector field actually is reducible to polynomial (provided it does not belong to the exceptional set). The proposition, however, cannot be applied to the classification of vector fields since it says nothing about the index of finite determinacy (the index $i(X)$ of finite determinacy of a vector field X is defined to be the minimal number k such that X is k-determined). One of the main objectives of the present paper is to find the index of finite determinacy of a given 1-resonant vector field.

The known results concerning the index of finite determinacy are as follows: Complete classification of 1-resonant vector fields is known only on $\mathbb{R}$ and $\mathbb{R}^2$ (see [6]). On $\mathbb{R}^n$, $n > 2$, only the so-called strongly 1-resonant vector fields (see the explaination shortly) are classified (see [2, 9]). In [7], the author studies *all the generic* vector fields on $\mathbb{R}^3$ (where the case that vector fields admit finitely many resonant relations is also discussed). In the present paper (see [8] also) we are going to give a complete classification of all (non-strongly) 1-resonant vector fields on $\mathbb{R}^3$ with arbitrary nonlinear parts. More precisely, given such a vector field X on $\mathbb{R}^3$, we shall find the index of finite determinacy $i(X)$, the number of moduli $\mu(X)$ which distinguishes closed vector fields of **Q** that are not smoothly conjugated, and the simplest normal form to which X can be reduced under a smooth change of coordinates. Our results reveal the intrinsic geometric relations between the central variable and the hyperbolic variables. Namely, given X with fixed degeneracy on the central manifol, we show that if X restricted to the hyperbolic manifold is not *too* degenerated (which is reflected in the number q in the following sections) then $i(X)$ depends on the central variable as well as on the hyperbolic ones. On the other hand, if X restricted to the hyperbolic manifold is *too* degenerated then $i(X)$ is totally determined by the central variable.

Now we characterize the eigenvalues of 1-resonant vector fields. Let X be such a vector field on $\mathbb{R}^3$ and its eigenvalues admit a relation

$$\alpha_1\lambda_1 + \alpha_2\lambda_2 + \alpha_3\lambda_3 = 0, \quad \alpha_i \in \mathbb{Z}_+, \quad |\alpha| > 0. \tag{3}$$

Then the following lemma holds.

Lemma 1.1. *For any 1-resonant vector field, either*

(i) all resonant relations $\lambda_i = k_1\lambda_1+k_2\lambda_2+k_3\lambda_3$ *are corollary of relation (3), i.e., there exists an integer* $l > 0$ *such that* $(k_1, k_2, k_3) = l(\alpha_1, \alpha_2, \alpha_3)+e_i$, *where* $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$ *and* $e_3 = (0, 0, 1)$; *or*

(ii) up to enumeration, the eigenvalues are $(0, \lambda_0, m\lambda_0)$, *where* $\lambda_0 \neq 0$ *and* m *is a natural integer.*

Vector fields of the first case in the lemma are called *strongly* 1-resonant.

We leave the proof of the lemma to the reader as an exercise. In this paper we consider vector fields of the second case of the lemma, that is, non-strongly 1-resonant vector fields. According to all possible values of m we have three cases: (1) $m \geq 2$; (2) $m = 1$ and the linear part of the vector fields is not diagonalizable; and (3) $m = 1$ and the linear part of the vector fields is diagonalizable.

The paper is organized as follows: In Section 2 we shall present the main results of the paper. Section 3 contains a brief exposition about normalization of vector fields. The proof of the results is given in Section 4.

2 Main Results

In this section we shall discuss all the three cases as specified above. Let X be a smooth vector field with eigenvalues $(0, \lambda, m\lambda)$. Then one sees that in all the three cases X has a 1-dimensional center manifold and the restricted normal form to the center manifold takes the form $\dot{x}_1 = f(x_1)$, where $f(x_1)$ is a formal power series, $f(0) = f'(0) = 0$. If $f(x_1)$ is not flat, then there exists an integer $p \geq 1$ such that $f(x_1) = x_1^{p+1} f_1(x_1)$, where $f_1(0) \neq 0$. In generic case, $p = 1$. It is clear that p is invariant of X due to the fact that if two vector fields are equivalent then their restrictions to the center manifold are equivalent.

Remark 2.2. Vector fields with flat $f(x_1)$ belong to the exceptional set **E** as described in Proposition 1.1.

Convention. We shall use capital letters $P_s(\cdot)$, $Q_s(\cdot)$, $R_s(\cdot)$, etc. to denote polynomials, where the subscript s stands for their degree.

2.1 Vector fields with eigenvalues $(0, \lambda, m\lambda)$, $m \geq 2$

The Poincare-Dulac resonant normal form in this case is given by

$$\dot{x}_1 = x_1^{p+1} f_1(x_1), \ \dot{x}_2 = x_2 f_2(x_1), \ \dot{x}_3 = x_3 f_3(x_1) + x_2^m f_4(x_1), \tag{1}$$

where $f_i(x_1)$ are formal series, $f_1(0) = a_p \neq 0$, $f_2(0) = \lambda$ and $f_3(0) = m\lambda$. Define an integer q by

$$f_3(x_1) - mf_2(x_1) = \tilde{a}_q x_1^q + \cdots, \quad \tilde{a}_q \neq 0 \tag{2}$$

(if $f_3(x_1) - mf_2(x_1) \equiv 0$ then $q = \infty$). We distinguish two cases:
Case **I**: $q \neq p$, or $q = p$ but $\tilde{a}_p/a_p \notin \{1, 2, \dots\}$.
Case **II**: $q = p$ and $\tilde{a}_p/a_p = k \in \{1, 2, \dots\}$.

Theorem 1. *Let X be a vector field with eigenvalues $(0, \lambda, m\lambda)$, $m \geq 2$. Then, in terms of (1) and (2),*
(1) $\mu(X) = 2p + 1$;
(2)

$$i(X) = \begin{cases} max(2p+1, m+s), \ \ where\ s = min(p, q-1) & \textit{Case } \mathbf{I} \\ max(2p+1, m+p+k) & \textit{Case } \mathbf{II} \end{cases}$$

(3) *The simplest normal form of X is as follows:*

$$\dot{x}_1 = \pm x_1^{p+1} + a x_1^{2p+1} \qquad \dot{x}_2 = x_2 P_p(x_1)$$

$$\dot{x}_3 = \begin{cases} m x_3 \tilde{P}_p(x_1) + x_2^m R_s(x_1) & \textit{Case } \mathbf{I} \\ m x_3 P_p(x_1) \pm k x_1^p x_3 + R_{p-1}(x_1) x_2^m + b x_1^{p+k} x_2^m & \textit{Case } \mathbf{II} \end{cases}$$

where $s = min(p, q-1)$, $\tilde{P}_p(x_1) - P_p(x_1) = \tilde{a}_q x_1^q + \cdots$, $\tilde{a}_q \neq 0$ if $q < p$, or $\tilde{a}_p \neq \pm 1, \pm 2, \dots$ if $p = q$, $R(0) \in \{0, 1\}$.

2.2 Vector fields with eigenvalues $(0, \lambda, \lambda)$ and non-diagonal linear part

In this case the Poincare-Dulac resonant normal form is as follows:

$$\begin{aligned} \dot{x}_1 &= x_1^{p+1} f_1(x_1), \\ \dot{x}_2 &= x_2 f_2(x_1) + x_3 f_3(x_1), \\ \dot{x}_3 &= x_2 f_4(x_1) + x_3 f_5(x_1), \end{aligned} \tag{3}$$

where $f_i(x_1)$ are formal series, $f_1(0) \neq 0$, $f_2(0) = f_5(0) = \lambda$, $f_3(0) = 1$, $f_4(0) = 0$. Define an integer q by

$$f_4(x_1) = c x_1^q + \cdots, \quad c \neq 0. \tag{4}$$

If $f_4(x_1)$ is the zero formal series then define $q = \infty$. We distinguish two cases:

Case **I**: $q \leq 2p$;
Case **II**: $q > 2p$.

Theorem 2. *If X is a vector field with eigenvalues* $(0, \lambda, \lambda)$ *and non-diagonal linear approximation, then in terms of* (1) *and* (2),

(1) $\mu(X) = \begin{cases} 2p+1 & \textit{Case } \mathbf{I} \\ p+1 & \textit{Case } \mathbf{II} \end{cases}$

(2) $i(X) = \begin{cases} max(2p+1, p+q+1) & \textit{Case } \mathbf{I} \\ 2p+1 & \textit{Case } \mathbf{II} \end{cases}$

(3) *X is smoothly conjugated to the following simplest normal forms:*

$$\dot{x}_1 = \pm x_1^{p+1} + a x_1^{2p+1} \qquad \dot{x}_2 = \lambda x_2 + x_3,$$

$$\dot{x}_3 = \begin{cases} x_3 P_p(x_1) + x_1^q Q_{p-1}(x_1) x_2 & \textit{Case } \mathbf{I} \\ x_3 P_p(x_1) & \textit{Case } \mathbf{II} \end{cases}$$

where $P_p(0) = \lambda$, $Q_{p-1}(0) \neq 0$.

2.3 Vector fields with eigenvalues $(0, \lambda, \lambda)$ and diagonalizable linear part

Notice that the diagonalizability means that $f_3(0) = 0$ in (1). The number p can be defined as in (1). To describe the normal form we need to introduce another integer q. In terms of (3), consider the matrix

$$M_r = \begin{pmatrix} f_2^{(r)}(0) & f_3^{(r)}(0) \\ f_4^{(r)}(0) & f_5^{(r)}(0) \end{pmatrix}, \quad r = 0, 1, 2, \dots \tag{5}$$

Denote by μ_r and ν_r the eigenvalues of M_r. The number q is defined to be the minimum number r such that $\mu_r \neq \nu_r$ or $\mu_r = \nu_r$ but M_r is not diagonal. In other words, the Jordan normal forms of the matrixes $M_0, \dots, M_{q-1}$ are of the form $diag(\alpha, \alpha)$, whereas the Jordan normal form of M_q takes one of the following forms:

$$\begin{pmatrix} \mu_q & 0 \\ 0 & \nu_q \end{pmatrix}, \qquad \begin{pmatrix} \mu & 1 \\ 0 & \mu \end{pmatrix}, \qquad \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}, \tag{6}$$

where $\mu_q - \nu_q \neq 0$, $\beta \neq 0$. If $q = p$ we also introduce

$$\tau = \frac{\nu_p - \mu_p}{f_1(0)}. \tag{7}$$

We shall distinguish the following cases:

Case **I**. $q < p$, or $q = p$ but $\tau \notin \mathbb{Z}$. It contains the following subcases:

case 1.1. M_q is equivalent to the first matrix in (6), or $q = p$ and $\tau \notin \mathbb{Z}$.
case 1.2. M_q is equivalent to the second matrix in (6),
case 1.3. M_q is equivalent to the third matrix in (6).

Case **II**: $q = p$ and $\tau = k \in \{1, 2, \dots\}$ (if $k \in \{-1, -2, \dots\}$, then enumerate x_2 and x_3).

Case **III**: $q > p$.

Theorem 3. *If X is a vector field with eigenvalues* $(0, \lambda, \lambda)$ *and diagonalizable linear approximation, then in terms of* (1) *and* (3),

$$(1)\ \mu(X) = \begin{cases} 2p - q + 2 & \text{Case 1.1} \\ 4p - 3q + 1 & \text{Case 1.2} \\ 4p - 3q + 2 & \text{Case 1.3} \\ p + 2 & \text{Case } \mathbf{II} \\ p + 1 & \text{Case } \mathbf{III} \end{cases}$$

$$(2)\ i(X) = \begin{cases} 2p + 1 & \text{Cases } \mathbf{I} \text{ and } \mathbf{III} \\ max(2p + 1, p + k + 1) & \text{Case } \mathbf{II} \end{cases}$$

(3) *A list of the simplest normal forms is as follows. In all the cases*

$$\dot{x}_1 = \pm x_1^{p+1} + a x_1^{2p+1},$$

and $\dot{x}_2$ *and* $\dot{x}_3$ *are given by*

Case 1.1 $\quad \dot{x}_2 = x_2 P_p(x_1) \qquad \dot{x}_3 = x_3 \tilde{P}_p(x_1),$

where $\tilde{P}_p(x) - P_p(x) = O(x^q)$.

Case 1.2 $\quad \dot{x}_2 = x_2 P_p(x_1) + x_3 Q_p(x_1) \quad \dot{x}_3 = x_3 \tilde{P}_p(x_1) + x_2 \tilde{Q}_p(x_1),$

where $\tilde{P}_p(x_1) - P_p(x_1) = o(x_1^q)$, $Q_p(x_1) = x_1^q + \cdots$, $\tilde{Q}_p(x_1) = o(x_1^q)$.

Case 1.3 $\quad \dot{x}_2 = x_2 P_p(x_1) + x_3 Q_p(x_1) \quad \dot{x}_3 = -x_2 \tilde{Q}_p(x_1) + x_3 \tilde{P}_p(x_1),$

where $\tilde{P}_p(x_1) - P_p(x_1) = o(x_1^q)$, $\tilde{Q}_p(x_1) - Q_p(x_1) = O(x_1^q)$.

Case **II** $\quad \dot{x}_2 = x_2 P_p(x_1) \qquad \dot{x}_3 = x_3 P_p(x_1) \pm k x_1^p x_3 + b x_1^{p+k} x_2,$

where a and b are parameters.

Case **III** $\quad \dot{x}_2 = x_2 P_p(x_1) \qquad \dot{x}_3 = x_3 P_p(x_1).$

3 Normalization

In this section we briefly recall some techniques of normalization of vector fields. More detailed explanation can be found in [2, 6] (in [8] there is also a brief account).

The starting point of normalizing a given vector field is the Poincare-Dulac normal form or the Belitskii normal form if there is a nilpotent part. For further normalization, one needs only to perform resonant transformations, namely, changes of coordinates whose linear part is the identity and the nonlinear part contains the resonant monomials (see[2]).

Since any 1-resonant vector field always admits infinitely many resonant relations, we find that very often it is more convenient to normalize the vector field *jet-by-jet*. More exactly, once the K-jet of the given system is normalized by polynomial resonant transformations, then, to normalize the $k+1$ jet, one can perform a higher order polynomial resonant transformation. In other words, the normalization consists of a series of polynomial transformations.

Denote by $j^l X$ the l-jet of X and call two vector fields X and $\tilde{X}$ l-jet equivalent if there is a diffeomorphism Φ such that $j^l \Phi_* X = j^l \tilde{X}$. We shall use the following lemmas. Please refer to [8] for the proof.

Lemma 3.2. *If two vector fields X and Y have identical k-jets and the equation $j^{k+1}[X, \varphi] = j^{k+1}(Y - X)$ has a solution φ with a vanishing 1-jet, where $[X, \varphi]$ denotes the Lie bracket of X and φ, then X and Y are $k+1$-jet equivalent.*

If for any $k < \infty$ X and Y are k-jet equivalent then they are formally equivalent.

If for any vector field Y such that $j^k Y = 0$ the equation $[X, \varphi] = Y$ has a solution φ with $j^1 \varphi = 0$ then X and $j^k X$ are formally equivalent.

From Lemma 3.2 one sees that to prove a vector field X is formally k-jet determined, it suffices to prove that for any vector field Y, $j^k Y = 0$, the equation $[X, \varphi] = Y$ has a solution φ with a vanishing 1-jet. Moreover, it suffices to prove the solvability of this equation where Y is any formal resonant (with respect to the tuple of eigenvalues of X) vector field with vanishing k-jet (see [8]). In other words, following statement holds:

Lemma 3.3. *Let X be a smooth vector field, $X(0) = 0$. If for any formal resonant (with respect to the tuple of eigenvalues of X) vector field Y, $j^k Y = 0$, the equation*

$$[j^k X, \varphi] = Y \tag{1}$$

has a formal solution φ, $j^1 \varphi = 0$, then X is smoothly k-determined.

Equation (1) is called the homological equation.

4 Proof of Theorems

The proof of each theorem consists of the following steps: (1) the $i(X)$-jet of a vector field X can be normalized to one of the normal forms listed in Theorems 1-3; (2) assume that the normalization of $j^{i(X)}X$ has been done, then we need to prove that the homological equation (1) is solvable for any resonant vector field Y with a vanishing $i(X)$-jet; (3) the vector fields $j^{i(X)-1}X$ is not equivalent to $j^{i(X)}X$, namely, X is not $i(X)-1$ jet determined. In the proof we shall sometimes put these steps into one.

Notice that in all the three cases the resonant normal form restricted to the center manifold is the same: $\dot{x}_1 = x_1^{p+1} f_1(x_1)$, $f_1(0) \neq 0$. By the results of [6] on the normalization of vector fields on $\mathbb{R}$, the restricted system is smoothly conjugated to $\dot{x}_1 = \pm x_1^{p+1} + a x_1^{2p+1}$, where a is a parameter and the signs $\pm$ can be put into $+$ for odd p. We assume in what follows that this step has been taken. That is, we have performed (a series of) changes of coordinates of the form $x_1 \to x_1 + \alpha x_1^k$, $k = 2, 3, \dots$ with the possible exception for the case $k = p + 1$ (the term $x_1^{2p+1}\partial_{x_1}$ is unremovable). In further simplification of hyperbolic variables, we shall make the best use of this extra freedom.

Remind that any transformation of the form $x_1 \to x_1 + \alpha x_1^k$ does not change the number q defined in (2), (4) and (5), and vice versa, any resonant transformation on the hyperbolic variables have no influence on the center manifold.

4.1 Proof of Theorem 1

Let X be a vector field having eigenvalues $(0, \lambda, m\lambda)$ and taking resonant normal form (1), where

$$f_1(x_1) = \pm 1 + a x_1^p.$$

We shall only consider the case $f_1(0) = 1$. The case $f_1(0) = -1$ can be discussed in the same way. Remind that the number q defined in (2) and the cases defined before Theorem 1.

Case I: Denote by

$$f_2(x_1) = \lambda + a_1 x_1 + \cdots + a_{q-1} x_1^{q-1} + a_q x_1^q + \cdots,$$

then

$$f_3(x_1) = m(\lambda + a_1 x_1 + \cdots + a_{q-1} x_1^{q-1}) + b_q x_1^q + \cdots,$$

where $b_q - m a_q = 0$. If $q = p$ we also assume that $b_p - m a_p \neq 1, 2, \dots$.

The normalization of $j^{i(X)}X$ means the elimination of three kinds of resonant terms:

$$x_1^{p+k}x_2\frac{\partial}{\partial x_2} \qquad x_1^{p+k}x_3\frac{\partial}{\partial x_3} \qquad x_1^l x_2^m\frac{\partial}{\partial x_3}, \tag{1}$$

where $k = 1, \ldots, i(X) - p - 1$ and $l = min(p, q-1) + 1, \ldots, i(X) - m$. We shall show that this can be done via three kinds of resonant transformations:

$$x \to id. + \alpha x_1^k x_2 \partial_{x_2}; \; x \to id. + \beta x_1^k x_3 \partial_{x_3}; \; x \to id. + \gamma x_1^s x_2^m \partial_{x_3}. \tag{2}$$

In fact, it is straightforward to show that the first two kinds of transformations keep the first two kinds of terms in $j^{p+1}X$ unchanged while they bring the following contribution to the higher order terms $(0, k\alpha x_1^{p+k}x_2, *)$ and $(0, 0, k\beta x_1^{p+k}x_2 + *)$, where $*$ means some resonant terms of the third kind of (1). Therefore by choosing suitable α and β we can eliminate the first two kinds of terms from the original vector field.

Observe that only the terms $x_2^m x_1^k \partial_{x_3}$, $k \leq i(X) - p - 1$, of the third kind are influenced by the first two kinds of transformations. However, it is clear that the influence entirely depends on the $i(X)$-jet of X, therefore the index of finite determinacy does make sense. Moreover, as shown below, some of these terms with higher degrees can be removed by the third kind of transformations in (2). Indeed, this transformation keeps the first two kinds of terms unchanged whereas it brings to the original vector field the following effect:

(a) if $q < p$, $((ma_q - b_q)\gamma x_1^{q+s} + o(x_1^{q+s}))x_2^m \partial_{x_3}$;

(b) if $q > p$, $(s\gamma x_1^{p+s} + o(x_1^{p+s}))x_2^m \partial_{x_3}$;

(c) if $q = p$, $((ma_p - b_p + s)\gamma x_1^{p+s} + o(x_1^{p+s}))x_2^m \partial_{x_3}$.

Therefore the elimination of these terms follows precisely from the assumption of the present case.

Case II: By similar arguments as applied above, we can normalize the $i(X)$-jet of X. The difference in this case is that if $b_p - ma_p = k$ then the term $x_1^{p+k}x_2^m\partial_{x_3}$ is unremovable. Therefore the index $i(X)$ in this case depends on the value of k which can be arbitrarily large.

From the above discussion it is easy to see that the homological equation (1) is always solvable for any vector field Y with a vanishing $i(X)$-jet. In fact, one only needs to perform changes of coordinates (2) where $k \geq i(X) - p$ and $l \geq i(X) - m + 1$.

The extra parameter α in the change of coordinate $x_1 \to x_1 + \alpha x_1^{p+1}$ yields no further simplification on the hyperbolic variables.

The coefficient of term $x_2^m \partial_{x_3}$ can always be put into 1 or 0 (via a linear scaling).

Theorem 1 follows from the above arguments. □

4.2 Proof of Theorem 2

Given a vector field of normal form (3), we assume that $f_1(x_1) = 1 + ax_1^p$. Due to the existence of the nilpotent part in the linear approximation of the vector field, we can apply the Belistkii theorem to reduce (3) to

$$\dot{x}_1 = x_1^{p+1} + ax_1^{2p+1}, \quad \dot{x}_2 = x_2 g(x_1) + x_3, \quad \dot{x}_3 = x_3 g(x_1) + x_2 h(x_1), \tag{3}$$

where $g(0) = \lambda$, and $h(x_1) = cx_1^q + \cdots$, $c \neq 0$.

In the following normalization, we shall first look for transformations which normalize (3) as well as preserve its form. In fact, any changes of coordinates of the form

$$(x_1, x_2, x_3) \to id. + x_1^k(0, \alpha x_2 + \beta x_3, \gamma x_2 + \theta x_3) \tag{4}$$

makes the following contribution to (3)

$$\begin{pmatrix} 0 \\ (\gamma - \beta h(x_1) - k\alpha P(x_1))x_2 + (\theta - \alpha - k\beta P(x_1))x_3) \\ ((\alpha - \theta)h(x_1) - k\gamma P(x_1))x_2 + (\beta h(x_1) - \gamma - k\theta P(x_1))x_3) \end{pmatrix}^t x_1^k$$

where $P(x_1) = x_1^p + ax_1^{2p}$, and t means the transpose. More generally, the change of coordinates

$$(x_1, x_2, x_3) \to id. + \sum_{k=1}^{\infty} x_1^k(0, \alpha_k x_2 + \beta_k x_3, \gamma_k x_2 + \theta_k x_3)$$

brings to (3) the following effect

$$\sum_{k=1}^{\infty} x_1^k \begin{pmatrix} 0 \\ (\gamma_k - \beta_k h - k\alpha_k P(x_1))x_2 + (\theta_k - \alpha_k - k\beta_k P(x_1))x_3) \\ ((\alpha_k - \theta_k)h - k\gamma_k P(x_1))x_2 + (\beta_k h - \gamma_k - k\theta_k P(x_1))x_3) \end{pmatrix}^t. \tag{5}$$

Rearrange the terms of (5) by their degrees, we have the following.

Case (i): $1 \leq q \leq p$.

$$\dot{x}_2 = * + x_2 h_1(x_1) + x_3 h_2(x_1), \quad \dot{x}_3 = * + x_2 h_3(x_1) + x_3 h_4(x_1), \tag{6}$$

where $*$ stands for the terms of the original vector field, and

$$h_1(x) = \gamma_1 x + \cdots + \gamma_q x^q + (\gamma_{q+1} - c\beta_1)x^{q+1} + \cdots + (\gamma_p - c\beta_{p-q})x^p + \sum_{k=1}^{\infty}(\gamma_{p+k} - c\beta_{p-q+k} - k\alpha_k)x^{p+k}$$

$$h_2(x) = (\theta_1 - \alpha_1)x + \cdots + (\theta_p - \alpha_p)x^p + \sum_{k=1}^{\infty}(\theta_{p+k} - \alpha_{p+k} - k\beta_k)x^{p+k}$$

$$h_3(x) = c(\alpha_1 - \theta_1)x + \cdots + c(\alpha_{p-q} - \theta_{p-q})x^p + \sum_{k=1}^{\infty}(c(\alpha_{p-q+k} - \theta_{p-q+k}) - k\gamma_k)x^{p+k}$$

$$h_4(x) = -\gamma_1 x - \cdots - \gamma_q x^q - (\gamma_{q+1} - c\beta_1)x^{q+1} - \cdots - (\gamma_p - c\beta_{p-q})x^p - \sum_{k=1}^{\infty}(\gamma_{p+k} - c\beta_{p-q+k} + k\theta_k)x^{p+k}$$

We show below that, by choosing suitable parameters $\alpha_k, \beta_k, \gamma_k, \theta_k,\ k = 1, 2, \ldots$, we can normalize X to the following form

$$\dot{x}_1 = x_1^{p+1} + ax_1^{2p+1}\ \dot{x}_2 = x_3 + x_2 P_p(x_1)\ \dot{x}_3 = x_3 P_p(x_1) + x_1^q Q_p(x_1)x_2. \quad (7)$$

To arrive at this form one needs to prove the solvability of the following linear equations:

$$\begin{aligned}
&\theta_k = \alpha_k, \quad k = 1, \ldots, p\\
&\theta_{p+k} - \alpha_{p+k} - k\beta_k = 0,\ k = 1, 2, \ldots\\
&\gamma_k = 0,\ k = 1, \ldots, q\\
&\gamma_{q+k} = c_q\beta_k,\ k = 1, \ldots, p-q\\
&\gamma_{p+k} - c_q\beta_{p-q+k} - k\alpha_k = A_k = -\gamma_{p+k} + c_q\beta_{p-q+k} - k\theta_k, \quad k = 1, 2, \ldots
\end{aligned}$$

for any number A_k.

It is a direct matter to check the solvability of these equations and we omit the proof here.

To prove that the index of finite determinacy makes sense in the theorem, one needs to show that the elimination of any terms with order higher than $i(X)$ has no influence on the $i(X)$-jet of X. It is not hard to see this point from the above equations (one can realize so by investigating the relations between the subscripts of these parameters).

The highest degree term in $Q_p(x_1)x_2\partial_{x_3}$ can be removed by the extra freedom from the change of coordinates $x_1 \to x_1 + \alpha x_1^{p+1}$.

The equivalence of normal form (7) and the normal form listed in the theorem is obvious.

Case (ii): $p < q \leq 2p$. The only difference in this case is the rearrangement of those linear systems. A similar discussion can be given and is omitted here.

Case (iii): $q > 2p$. In this case, the functions in (5) are given by

$$\begin{aligned} h_1(x) &= \gamma_1 x + \cdots + \gamma_p x^p \\ &+ (\gamma_{p+1} - \alpha_1)x^{p+1} + \cdots + (\gamma_q - (q-p)\alpha_{q-p})x^q \\ &+ \sum_{k=1}^{\infty}(\gamma_{q+k} - c\beta_k - (q-p+k)\alpha_{q-p+k})x^{q+k} \end{aligned}$$

h_2 remains the same.

$$\begin{aligned} h_3(x) &= -\gamma_1 x^{p+1} - \cdots - (q-p)\gamma_{q-p}x^q \\ &+ \sum_{k=1}^{\infty}(c(\alpha_k - \theta_k) - (q-p+k)\gamma_{q-p+k})x^{q+k} \end{aligned}$$

$$\begin{aligned} h_4(x) &= -\gamma_1 x - \cdots - \gamma_p x^p \\ &- (\gamma_{p+1} + \theta_1)x^{p+1} - \cdots - (\gamma_q + (q-p)\theta_{q-p})x^q \\ &- \sum_{k=1}^{\infty}(\gamma_{q+k} - c\beta_k + (q-p+k)\theta_{q-p+k})x^{q+k} \end{aligned}$$

The corresponding normalization means the solvability of the following linear equations.

$$\begin{aligned} &\theta_k = \alpha_k, \quad k = 1, \ldots, p, \\ &\theta_{p+k} - \alpha_{p+k} = k\beta_k, \quad k = 1, 2, \ldots \\ &\gamma_k = 0, \quad k = 1, \ldots, p, \\ &\gamma_{p+k} - k\alpha_k = A_k = -\gamma_{p+k} - k\theta_k, \quad k = 1, \ldots, q-p \\ &\gamma_{q+k} - (q-p+k)\alpha_{q-p+k} - c\beta_k = A_{q-p+k} = -\gamma_{q+k} - (q-p+k)\theta_{q-p+k} + c\beta_k \end{aligned}$$

for any number A_k. We shall skip the details here.

The solvability of these equations means that the original vector field can be reduced to the following polynomial form

$$\dot{x}_1 = x_1^{p+1} + ax_1^{2p+1} \; \dot{x}_2 = x_3 + x_2 P_p(x_1) \; \dot{x}_3 = x_3 P_p(x_1),$$

which, like the previous cases, is equivalent to the final normal form given in the theorem.

Collecting all the facts above, we prove the theorem. □

Proof of Theorem 3. We shall only sketch the proof of the case 1.1. All the other cases can be treated similarly.

First, it is easy to see that this case implies that the functions in (3) take the following forms:

$$f_2(x_1) = \lambda + \mu_1 x_1 + \cdots + \mu_{q-1} x_1^{q-1} + \mu_q x_1^q + \dots$$
$$f_3(x_1) = O(x_1^{q+1}), \quad f_4(x_1) = O(x_1^{q+1})$$
$$f_5(x_1) = \lambda + \mu_1 x_1 + \cdots + \mu_{q-1} x_1^{q-1} + \nu_q x_1^q + \dots$$

where $\nu_q - \mu_q \neq 0$, and if $p = q$ we also assume that $\nu_p - \mu_p \neq 1, 2, \dots$ We shall show that all the terms in f_3 and f_4 and the those terms in f_2 and f_5 whose degrees are higher than p are removable. In fact, one can check that the change of coordinates $x \to id. + (0, \beta_1 x_1 x_3, \gamma_1 x_1 x_2)$ makes the following effect to the $q+1$-jet of the original vector field $(\mu_q - \nu_q) x_1^{q+1} (\beta_1 x_3 \partial_{x_2} - \gamma_1 x_2 \partial_{x_3})$. Therefore by choosing suitable β_1 and γ_1, one can remove the terms $A x_1^{q+1} x_3 \partial_{x_2} + B x_1^{q+1} x_2 \partial_{x_3}$ for any parameters A and B. Notice that the transformed vector field admits the same resonant form. By performing a series of this kind of transformations (with higher order terms), we can remove, jet-by-jet, all the terms in f_3 and f_4. The normalization of f_2 and f_5 can also be fulfilled jet by jet. To this end, take transformation $x \to id. + (0, \alpha_1 x_1 x_2, \theta_1 x_1 x_3)$. Then it keeps the $p+1$-jet of X unchanged and brings the terms $x_1^{p+1}(\alpha_1 x_2 \partial_{x_2} + \theta_1 x_3 \partial_{x_3})$ to the $p+2$-jet. Therefore with suitable α_1 and θ_1, one can normalize the $p+2$-jet. Carrying on this process with higher order transformations, one can arrive at the normal form given in the theorem. □

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